

作业一

1. 用换元积分法计算下列各题。

(1)

$$\begin{aligned} \int_1^2 \frac{dx}{(3x-1)^2} &\stackrel{u=3x-1}{=} \int_2^5 \frac{1}{3} \cdot \frac{du}{u^2} \\ &= \frac{1}{3}(-u^{-1}) \Big|_2^5 \\ &= \frac{1}{3} \left(-\frac{1}{5} + \frac{1}{2} \right) \\ &= \frac{1}{3} \cdot \frac{3}{10} \\ &= \frac{1}{10}. \end{aligned}$$

(3)

$$\begin{aligned} \int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx &\stackrel{u=\sqrt{x}-1}{=} \int_{\sqrt{4}-1}^{\sqrt{9}-1} \frac{u+1}{u} du(u+1)^2 \\ &= \int_1^2 \frac{u+1}{u} 2(u+1) du \\ &= 2 \int_1^2 \frac{u^2+2u+1}{u} du \\ &= 2 \int_1^2 u+2+\frac{1}{u} du \\ &= 2 \left(u^2/2 + 2u + \ln u \right) \Big|_1^2 \\ &= 2 \left[(2^2/2 + 2 \cdot 2 + \ln 2) - (1^2/2 + 2 \cdot 1 + \ln 1) \right] \\ &= 7 + 2 \ln 2. \end{aligned}$$

(9)

$$\begin{aligned}
\int_0^\pi \sqrt{\sin x - \sin^3 x} dx &= \int_0^\pi \sqrt{\sin x(1 - \sin^2 x)} dx \\
&= \int_0^\pi \sqrt{\sin x} \cdot \sqrt{(1 - \sin^2 x)} dx \\
&= \int_0^\pi \sqrt{\sin x} \cdot \sqrt{\cos^2 x} dx \\
&= \int_0^\pi \sqrt{\sin x} \cdot |\cos x| dx \\
&= \int_0^{\pi/2} \sqrt{\sin x} \cdot \cos x dx - \int_{\pi/2}^\pi \sqrt{\sin x} \cdot \cos x dx \\
&= \int_0^{\pi/2} \sqrt{\sin x} d\sin x - \int_{\pi/2}^\pi \sqrt{\sin x} d\sin x \\
&\stackrel{u=\sin x}{=} \int_0^1 \sqrt{u} du - \int_1^0 \sqrt{u} du \\
&= \int_0^1 \sqrt{u} du + \int_0^1 \sqrt{u} du \\
&= 2 \int_0^1 \sqrt{u} du \\
&= 2 \cdot \frac{2}{3} u^{3/2} \Big|_0^1 \\
&= \frac{4}{3}
\end{aligned}$$

(10)

$$\begin{aligned}
\int_{-1}^1 \frac{dx}{1 + \sqrt{1 - x^2}} &\stackrel{u = \arcsin x}{=} \int_{-\pi/2}^{\pi/2} \frac{d \sin u}{1 + \sqrt{1 - \sin^2 u}} \\
&= \int_{-\pi/2}^{\pi/2} \frac{\cos u du}{1 + \sqrt{\cos^2 u}} \\
&= \int_{-\pi/2}^{\pi/2} \frac{\cos u du}{1 + |\cos u|} \\
&= \int_{-\pi/2}^{\pi/2} \frac{\cos u du}{1 + \cos u} \\
&= 2 \int_0^{\pi/2} \frac{\cos u}{1 + \cos u} du \\
&= 2 \int_0^{\pi/2} \frac{1 + \cos u - 1}{1 + \cos u} du \\
&= 2 \int_0^{\pi/2} 1 - \frac{1}{1 + \cos u} du \\
&= 2 \int_0^{\pi/2} 1 - \frac{1}{2 \cos^2 \frac{u}{2}} du \\
&= 2 \int_0^{\pi/2} 1 du - \int_0^{\pi/2} \sec^2 \frac{u}{2} du \\
&= 2 \cdot \pi/2 - 2 \tan \frac{u}{2} \Big|_0^{\pi/2} \\
&= \pi - 2.
\end{aligned}$$

2.

(1)

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} x \cos 2x \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{4}} x \, d(\sin 2x) \\
&= \frac{1}{2} \left[x \sin 2x \Big|_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \sin 2x \, dx \right] \\
&= \frac{1}{2} \left[\frac{\pi}{4} \sin \frac{\pi}{2} - 0 \cdot \sin 0 + \frac{\cos 2x}{2} \Big|_0^{\frac{\pi}{4}} \right] \\
&= \frac{1}{2} \left[\frac{\pi}{4} + \frac{\cos \frac{\pi}{2}}{2} - \frac{\cos 0}{2} \right] \\
&= \frac{1}{2} \left[\frac{\pi}{4} + \frac{\cos \frac{\pi}{2}}{2} - \frac{\cos 0}{2} \right] \\
&= \frac{1}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) \\
&= \frac{\pi}{8} - \frac{1}{4}.
\end{aligned}$$

(2)

$$\begin{aligned}
\int_0^{e-1} (1+x) \ln^2(1+x) \, dx &\stackrel{u=1+x}{=} \int_1^e u \ln^2 u \, du \\
&= \int_1^e \ln^2 u \, d\frac{u^2}{2} \\
&= \frac{u^2}{2} \ln^2 u \Big|_1^e - \int_1^e \frac{u^2}{2} \, d\ln^2 u \\
&= \frac{e^2}{2} - \int_1^e u \ln u \, du \\
&= \frac{e^2}{2} - \int_1^e \ln u \, d\frac{u^2}{2} \\
&= \frac{e^2}{2} - \frac{u^2}{2} \ln u \Big|_1^e + \int_1^e \frac{u^2}{2} \, d\ln u \\
&= \frac{e^2}{2} - \frac{e^2}{2} + \int_1^e \frac{u}{2} \, du \\
&= \frac{u^2}{4} \Big|_1^e \\
&= \frac{e^2 - 1}{4}.
\end{aligned}$$

(3)

$$\begin{aligned}
\int_0^{\pi/2} e^{-x} \sin 2x \, dx &= - \int_0^{\pi/2} \sin 2x \, d(e^{-x}) \\
&= - \left[(\sin 2x)e^{-x} \Big|_0^{\pi/2} - \int_0^{\pi/2} e^{-x} \, d(\sin 2x) \right] \\
&= - \left[(\sin 2x)e^{-x} \Big|_0^{\pi/2} - 2 \int_0^{\pi/2} e^{-x} \cos 2x \, dx \right] \\
&= -(\sin 2x)e^{-x} \Big|_0^{\pi/2} + 2 \int_0^{\pi/2} e^{-x} \cos 2x \, dx \\
&= -(\sin 2x)e^{-x} \Big|_0^{\pi/2} - 2 \int_0^{\pi/2} \cos 2x \, d(e^{-x}) \\
&= -(\sin 2x)e^{-x} \Big|_0^{\pi/2} - 2 \left[(\cos 2x)e^{-x} \Big|_0^{\pi/2} - \int_0^{\pi/2} e^{-x} \, d(\cos 2x) \right] \\
&= -(\sin 2x)e^{-x} \Big|_0^{\pi/2} - 2 \left[(\cos 2x)e^{-x} \Big|_0^{\pi/2} + 2 \int_0^{\pi/2} e^{-x} \sin 2x \, dx \right] \\
&= \left[-(\sin 2x)e^{-x} - 2(\cos 2x)e^{-x} \right] \Big|_0^{\pi/2} - 4 \int_0^{\pi/2} e^{-x} \sin 2x \, dx.
\end{aligned}$$

因此

$$\begin{aligned}
\int_0^{\pi/2} e^{-x} \sin 2x \, dx &= \frac{1}{5} \left[-(\sin 2x)e^{-x} - 2(\cos 2x)e^{-x} \right] \Big|_0^{\pi/2} \\
&= \frac{2e^{-\pi/2}}{5} + \frac{2}{5}.
\end{aligned}$$

(4)

$$\begin{aligned}
\int x^3 e^{-x^2} \, dx &= \frac{1}{2} \int x^2 e^{-x^2} \, dx^2 \\
&\stackrel{u=x^2}{=} \frac{1}{2} \int u e^{-u} \, du \\
&= -\frac{1}{2} \int u \, de^{-u} \\
&= -\frac{1}{2} \left[ue^{-u} - \int e^{-u} \, du \right] \\
&= -\frac{1}{2} [ue^{-u} + e^{-u}] + C \\
&\stackrel{u=x^2}{=} -\frac{1}{2} [x^2 e^{-x^2} + e^{-x^2}] + C.
\end{aligned}$$

根据微积分基本定理

$$\begin{aligned}
\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2} \, dx &= -\frac{1}{2} \left[x^2 e^{-x^2} + e^{-x^2} \right] \Big|_0^{\sqrt{\ln 2}} \\
&= -\frac{1}{2} [\ln 2 \cdot e^{-\ln 2} + e^{-\ln 2} - 0 - e^0] \\
&= -\frac{1}{2} \left[\frac{\ln 2}{2} + \frac{1}{2} - 1 \right] \\
&= \frac{1}{4} - \frac{\ln 2}{4}.
\end{aligned}$$